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Optimization of complex functions and the algorithm for exact geometric search for complex roots of a polynomial

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Abstract: The paper describes an application for visualization of four-dimensional graphs of a complex variable function. This application allowed us to construct an exact geometric algorithm for finding the real and complex roots of a polynomial on the same plane. A graph of an n-th order polynomial on the real plane allows us to define geometrically all the real roots. Number of real roots varies from 0 to n. The rest of the roots are complex and not determined by the graph. In the article, in addition to the graph of the basic polynomial, two auxiliary graphs are constructed, which allow us to represent all complex roots on the same real plane. Realization of this method is considered in detail for the solution of a cubic polynomial. In this case the method has exceptional features in comparison with polynomials of other degrees. We also propose an algorithm for constructing auxiliary functions for the general case of a polynomial of order n which have exact formulas for polynomials with order $n \leq 10$. The algorithm for the first time builds the exact hodograph of poles for the control systems with feedback. We generalize the concepts of stationary and extremal points to the case of a complex function. The absence of the possibility of comparing the complex values of the objective function is compensated by an analysis of the behavior of the stationary point under small perturbations of the polynomial by linear functions. Optimality criteria are proposed using complex trajectories of stationary points.

Keywords: complex function, polynom, roots, graph , exact algorythm, root hodograph, stationary point, extremal point, optimality criterion

1. INTRODUCTION

Complex numbers were introduced as an auxiliary tool for finding the real roots of polynomials 3 rd and 4 th orders. The basic algebraic theorem proves that the number of complex and real roots is equal to order of the polynomial. The real roots are easy to find and illustrated on the graph. Complex roots do not possess this property. There are several exact algebraic and iterative numerical methods for finding the roots (Zaguskin (1988)). Precise methods for polynomials with arbitrary coefficients, as is known, exist only for polynomials of order $n \leq 4$. These are the well-known Cardano ($n = 3$) and Torriceli ($n = 4$) formulas.

In this article, the exact geometric method is understood as the use of graphs of functions that have an exact analytical expression, and a one-sided ruler with the ability to draw parallel lines. Along with the graph of the original polynomial $f(x)$, we construct a graph of auxiliary multivalued "conjugate" function $f_S(x)$. On this graph, a subset is distinguished, which is called the gluing set and is denoted S_f. Then the second auxiliary multivalued "carrier" function $f_N(x)$ is constructed. The domain of definition of $f_N(x)$ coincides with the projection of the gluing set S_f onto the axis Ox. To find the complex roots of the equation $f(x) = \Delta$, where Δ is an arbitrary real number, we use the same real plane. The Ox axis is associated with the real part of the complex number, and the Oy axis is joined with the imaginary part. Both auxiliary functions $f_S(x)$ and $f_N(x)$ have an exact analytic notation for $n \leq 10$. For polynomials of higher degrees, the graphs of the auxiliary functions must be constructed numerically. The algorithm assumes drawing the straight lines. First, we draw a horizontal line with the equation $y = \Delta$ and find the intersection points with the glueing set. Then through each such point, we draw a vertical line and find the points of intersection with the carrier graph. These last points are complex roots of the equation $f(x) = \Delta$. When the free coefficient Δ changes, the method shows the migration way [8], in which the roots move. We can see how a real root splits into two complex ones, and vice versa, two complex conjugate roots join into one real root. The geometric method for solving the cubic equation is studied in detail in the article. In this case, the method has exceptional features in comparison with polynomials of other degrees, namely, the auxiliary functions are expressed in terms of the original polynomial $f(x)$. It is shown that the conjugate function $f_S(x) = f(-2x - a_1/a_0)$, and the carrier function $f_N(x)$ is expressed in terms of the derivative of the original polynomial $f_N(x) = \sqrt{f'(x)}$. Moreover, the conjugate function is a single-valued function, while for other degrees it is multivalued.

The developed method can be used, for example, for an exact geometric representation of the root locus in the study of control systems with feedback, in which there

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is a parametric block multiplier k. The parameter k plays the role of the free coefficient Δ and the problem is to find all the poles of the characteristic polynomial of a closed system (Uderman (1972) —Bendrikov (1955)).

2. GRAPH OF A FUNCTION OF A COMPLEX VARIABLE

It is required to construct a graph of a complex function $y = f(z)$:

$$u + vi = f(a + bi). \tag{1}$$

The point of the graph is determined by four components u, v, a, b. We represent the graph G of a function in a three-dimensional coordinate system of variables (Trofimov (2017)). The parameters u and v form two coordinate axes and define the complex plane $\mathbb{C}$ of the values of the function f. The third axis is for the parameter a. In the plane $a = a_0$ we construct the intersection of this plane and the graph G. This intersection is a parametric graph of the function $u + vi = f(a_0 + bi)$. The number b plays the role of a parameter (Fig. 1).

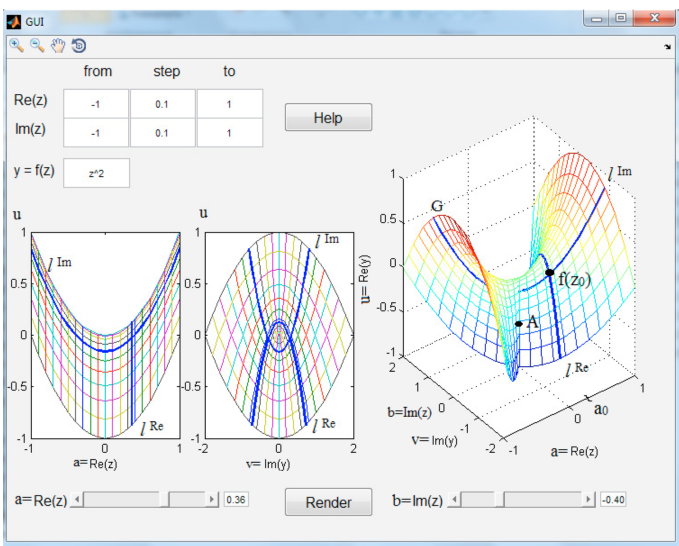


Fig. 1. Interface of application for constructing graph of complex functions on example $y = z^2$

Thus, the four-dimensional graph of a complex function (1) is represented with help of three Cartesian coordinates $\{u, v, a\}$ and one parametric coordinate b. The four-dimensional graph has the form of a surface G in the three-dimensional coordinate system u, v, a. Let us denote the current point $z_0 = a_0 + b_0 i$. The image of this point is represented as the intersection point of the lines $l^{Im}_{b_0} = \{a + b_0 \cdot i : a \in \mathbb{R}\}$ and $l^{Re}_{a_0} = \{a_0 + b \cdot i : b \in \mathbb{R}\}$. The graph depicts a grid of images of lines l^{Im}_a, l^{Re}_b, where a and b change with a given step.

The current point z_0 is specified by the control elements of the "slider" type. 3D-rotations and scaling of the graphics are performed by standard Matlab tools. The rectangular area of the domain definition and the sampling step are implemented by the user in the interface. For a four-dimensional graph, six projections onto two-dimensional planes are possible. Two similar projections are depicted in the Fig.1 . The first projection of G onto the plane $\{a, u\}$ allows to see the real function $u = f(a)$ of real argument a if $b_0 = 0$. The second projection of G onto the plane $\{v, u\}$ shows parametric complex-valued function $u + vi = f(a_0 + bi)$ at a fixed value of a_0 and with a parameter b.

Graph G contains a grid of lines l^{Im}, l^{Re}. Projection of this grid contains false points of intersection. As a result, two-dimensional projections of the grid give a distorted representation of the function graph G (Fig. 1). This explains the difficulty in drawing complex graphs in similar applications.

3. THE MAIN THEOREM OF ARITHMETIC

As is known, the polynomial $f(x)$ of order n has exactly n real or complex roots, possibly multiple. Our application clearly illustrates this theorem. The graph G of the polynomial $f(z)$ of the complex variable $z = a + bi$ has the form of a self-intersecting surface consisting of n leaves. The graph shows that the line $l = \{(u_0, v_0, a) : a \in \mathbb{R}\}$ crosses each sheet at one point. Therefore, the equation $f(z) = u_0 + v_0 i$ has exactly n roots. It is seen that if $v_0 \neq 0$, then among the complex roots there are no conjugate numbers. If $v_0 = 0$, then the line l is in the coordinate plane $\{a, u\}$ and intersects the graph of G at the point $A = (a_0, u_0, 0)$. Suppose that the point A lies on the self-intersection of the surface G. If we modify the complex part b of the argument z in the application, then the graph shows that self-intersection at the point A is possible for some pair of complex conjugate numbers $z_{1,2} = a_0 \pm bi$. The numbers $z_{1,2}$ are the roots of the equation $f(z) = u_0$. Thus, the graph of a complex polynomial not only allows us to illustrate the main theorem of arithmetic, but also to approach the algorithm for finding complex roots of polynomials.

4. BASIC CONCEPTS

Proposition 1. Let $f(x)$ be a polynomial of order n with real coefficients. The gluing set of $f(x)$ is called the set S_f of points (x, y) of the real plane $\mathbb{R}^2$, for which there are distinct complex mutually conjugate numbers z_1 and z_2 such that $Re\ z_1 = Re\ z_2 = x$ and $f(z_1) = f(z_2) = y$.

The gluing set is the set of intersection points (or gluing) of the four-dimensional graph of the complex function $f(z)$ of the complex variable z (Trofimov (2017)). One can easy prove next

Lemma 2. Let $f(x)$ be a polynomial of the third order. If the point $(a, \Delta) \in S_f$, then the equation

$$f(x) = \Delta \tag{2}$$

has one real and two different complex conjugate roots, and the real part of the complex roots is equal to a.

Lemma 3. Let $f(x) = a_0 x^3 + a_1 x^2 + a_2 x + a_3$ be a polynomial of the third order with real coefficients, $a_0 \neq 0$. Then the gluing set S_f of the polynomial $f(x)$ lies on the graph of the function $f_s(t)$, where $f_s(t) = f(-2t - a_1/a_0)$.

The function $f_s(t)$ is said to be conjugate to the function $f(t)$.

Proof. Obviously, $f_s(t)$ is also a polynomial of the third order. Take an arbitrary point $(x_0, \Delta) \in S_f$ and construct the equation (2). By Lemma 2 , equation (2) has exactly one real root. Denote this real root $z_1(\Delta)$ and find other two complex roots z_2 and z_3. Obviously, numbers z_2 and z_3 are the roots of the polynomial of the second order

$$\frac{f(x) - \Delta}{x - z_1} = A \cdot x^2 + B \cdot x + C,$$

where $A = a_0, B = a_0 z_1 + a_1, C = a_0 z_1^2 + a_1 z_1 + a_2$.

Equation $Az^2 + Bz + C = 0$ has two roots $z_{2,3} = -\frac{B}{2A} \pm \frac{\sqrt{D}}{2A}$, where the discriminant $D = B^2 - 4 \cdot A \cdot C$ depends on z_1 and hence on Δ.

Thus, equation (2) has three roots: one real z_1 and two complex conjugated ones

$$z_2 = -\frac{z_1}{2} - \frac{a_1}{2a_0} + \frac{\sqrt{D}}{2A}, z_3 = -\frac{z_1}{2} - \frac{a_1}{2a_0} - \frac{\sqrt{D}}{2A}. \quad (3)$$

By the definition of the set S_f, for the point $(x_0, \Delta) \in S_f$ there exist different complex conjugate numbers z_2^0 and z_3^0 such that $f(z_2^0) = f(z_3^0) = \Delta$ and $Re z_2^0 = Re z_3^0 = x_0$.

It follows that z_2^0 and z_3^0 are complex roots of equation (2) and, in particular, they differ from the real root z_1. Therefore, the roots z_2 and z_3 coincide with the numbers z_2^0 and z_3^0. So, if $(x_0, \Delta) \in S_f$, then

$$x_0 = -\frac{z_1(\Delta)}{2} - \frac{a_1}{2a_0}. \quad (4)$$

The set S_f has the single-valued property: if (x_0, Δ_1) and (x_0, Δ_2) belong to S_f, then $\Delta_1 = \Delta_2$. Indeed, from (4) we obtain

$$z_1(\Delta_1) = z_1(\Delta_2) = -2x_0 - \frac{a_1}{a_0}.$$

Since $z_1(\Delta_1)$ and $z_1(\Delta_2)$ are the roots of the corresponding equation (2), then $\Delta_1 = f(z_1(\Delta_2)) = f(z_1(\Delta_2)) = \Delta_2$. Thus, the set S_f lies on the graph of a single-valued conjugate function $y = f_s(x)$, and $f_s(x_0) = \Delta$, that is,

$$f_s\left(-\frac{z_1(\Delta)}{2} - \frac{a_1}{2a_0}\right) = \Delta. \quad (5)$$

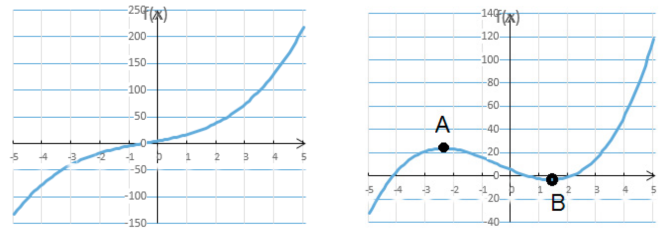
Replace $t = -\frac{z_1(\Delta)}{2} - \frac{a_1}{2a_0}$. Then $z_1(\Delta) = -2t - \frac{a_1}{a_0}$. Since z_1 is the root of (2), then

$$f\left(-2t - \frac{a_1}{a_0}\right) = f(z_1(\Delta)) = \Delta. \quad (6)$$

Comparing relations (5) and (6), we obtain $f_s(t) = f(-2t - \frac{a_1}{a_0})$. Lemma 2 is proved.

The graph of the conjugate function $f_s(x)$ is formed from the graph of the cubic parabola $f(x)$ in the following way. We compress the graph $f(x)$ twice with respect to the vertical axis passing through the inflection point and turn it around the same axis.

Lemma 4. Let $f(x)$ be a cubic polynomial with real coefficients. Then the graph of the conjugate function $f_s(x)$ passes through points of local extrema $f(x)$, if they exist. In addition, the unique inflection points of the functions f and f_s coincide.



a) b)

Fig. 2. Two types of graphs of a cubic polynomial: a) without local extrema, b) with two local extrema

Proof. Assume that x^* is a local extremum point of $f(x)$. One can find x^* from the equation $f'(x) = 0$ and verify the equality $f_s(x^*) = 0$ by a direct substitution. We simplify the calculations by changing the variable.

It is known that with parallel shifts the shape of the function graph does not change. Let us shift the graphs of the functions $f(x)$ and $f_s(x)$: horizontally by $r = -\frac{a_1}{3a_0}$ to the left using the substitution $x = u + r = u + (-\frac{a_1}{3a_0})$, and vertically down by a_3 .

We obtain $g(u) = f(u + (-\frac{a_1}{3a_0})) - a_3$. The function $g(u)$ is a cubic parabola, for which the inflection point coincides with the coordinates origin. This easily implies that $g(u) = A \cdot u^3 + C \cdot u$. Obviously, the function $g(u)$ is odd and passes through the coordinates origin.

It suffices to prove Lemma 3 for the function $g(u)$. We assume that $A > 0$ and suppose there are local extremum points for g. Then they have the form $u_{1,2} = \pm\sqrt{\frac{-C}{3A}}$, from which, in particular, $C < 0$. Substituting, for example, u_1 into the functions $g(u)$ and $g_s(u)$, we obtain

$$g(u_1) = g_s(u_1) = \frac{2}{3} C \sqrt{-\frac{C}{3A}}. \quad (7)$$

Thus, the extremal points of the graph $g(u)$ lie on the graph of its conjugate function $g_s(u)$.

It is obvious that the inflection point of the function $g_s(u) = -8u^3 - 2Cu$ coincides with the inflection point of the function $g(u)$. Lemma 3 is proved.

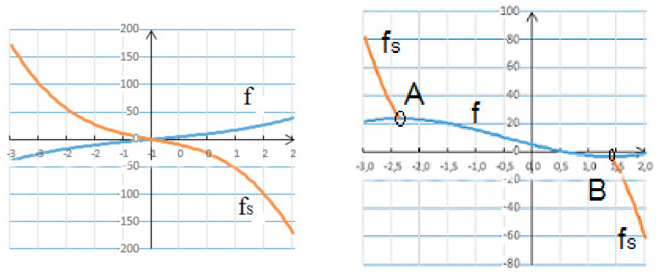
5. CONSTRUCTION OF A GLUING SET FOR A POLYNOMIAL OF ORDER 3

Consider the polynomial $f(x) = a_0 x^3 + a_1 x^2 + a_2 x + a_3$, where $a_0 > 0$. The graph of a cubic polynomial $f(x)$ can be of two types (Fig. 2):

By Lemma 4, the graph of the conjugate function $f_s(x)$ passes through the extremum points and the inflection point of the function $f(x)$.

We construct the gluing set S_f, which by Lemma 3 is contained in the graph of the conjugate function $f_s(x)$.

The type of the cubic parabola (Fig. 2) depends on the sign of the discriminant of the cubic polynomial $D = 4(a_1^2 - 3a_0 \cdot a_2)$. Indeed, the abscissas of the points of local extrema of the function $f(x)$ are real roots of its derivative $f'(x) = 3a_0 x^2 + 2a_1 x + a_2$.



a) b)

Fig. 3. Two types of gluing set for a third-order polynomial: a) without local extrema, b) with local extrema.

The discriminant of the quadratic equation $f'(x) = 0$, which determines the presence of real roots of $f(x)$, coincides with D.

If $D \leq 0$, then $f(x)$ does not have local extrema (Fig. 2a) and by Lemma 3 for any Δ the equation $f(x) = \Delta$ has one real and two complex roots. Therefore, gluing set S_f coincides with the entire graph of the conjugate function $f_s(x)$ (Fig. 2a).

Let $D > 0$. Then the graph of $f(x)$ has the form Fig 2b), where $A(x_1, y_1)$ and $B(x_2, y_2)$ are extreme points of the graph, and

$$y_1 > y_2, x_{1,2} = \frac{-2a_1 \pm \sqrt{D}}{6} \tag{8}$$

Take $\Delta_0 \in [y_2, y_1]$. Then the cubic equation $f(x) = \Delta_0$ has three real roots. If $\Delta_0 = y_2$ or $\Delta_0 = y_1$, then two of them are multiple. Therefore, by Lemma 2, for any x the point $(x, \Delta_0) \notin S_f$. Consequently, the horizontal line $y = \Delta_0$ passing through the point $(0, \Delta_0)$ does not intersect the set S_f. Thus, gluing set S_f is formed from the graph of the conjugate function $f_s(x)$ by removing the closed fragment of the graph from A to B (Fig 3b).

Thus, on the same plane $\mathbb{R}^2$ three sets are constructed: the graph of the initial polynomial $f(x)$, the graph of the conjugate function $f_s(x)$, and the gluing set S_f, which is contained in second set.

6. MAIN RESULTS

We consider the cubic equation

$$f(x) \equiv x^3 + a_1x^2 + a_2x + a_3 = \Delta \tag{9}$$

for an arbitrary real Δ. We assume that $a_0 = 1$. Otherwise, the final results of this section can easily be modified by replacing the coefficients a_k by $\frac{a_k}{a_0}$ for $k = 1, 2, 3$. The real roots of equation (9) can be found from the graph of the original polynomial. We shall seek the complex roots of this equation in the form $x = a + bi, b \neq 0$.

Substituting x into (9), we obtain

$$f(a+bi) = [a^3 - 3ab^2 + a_1a^2 - a_1b^2 + a_2a + a_3] + [3a^2b- \\ -b^3 + 2a_1ab + a_2b]i = R(a,b) + I(a,b)\cdot i = \Delta + 0\cdot i.$$

Equate to zero the imaginary part of $I(a, b)$:

$$3a^2b - b^3 + 2a_1ab + a_2b = 0.$$

Dividing by $b\neq0$, we obtain $3a^2 - b^2 + 2a_1a + a_2 = 0$, whence

$$b^2 = 3a^2 + 2a_1a + a_2. \tag{10}$$

Transform the real part $R(a, b)$ of the number $f(a + bi)$

$$\begin{aligned} R(a,b) &= [a^3 + a_1a^2 + a_2a + a_3] - [3a + a_1]\cdot b^2 \\ &= f(-2a - \frac{a_1}{a_0}) = f_s(a) = \Delta. \end{aligned} \tag{11}$$

Thus, it is proved

Lemma 5. If the complex number $x = a + bi$ is a root of the equation (9), then

$$f_s(a) = \Delta. \tag{12}$$

From Fig. 3 we can see, that under the conditions of Lemma 5 the function f_s is invertible. Therefore, equality (12) can be written in the form

$$a = f_s^{-1}(\Delta). \tag{13}$$

Relation (13) gives a geometric algorithm of finding the real part a of the complex root $x = a + bi$ of the equation $f(x) = \Delta$. Now we show how to find the complex component b of the root x.

Proposition 6. The carrier set N_f of a polynomial $f(x)$ is defined to be the set of all complex numbers u with nonzero complex part, for which $f(u)$ is a real number, that is

$$N_f = \{u \in \mathbb{C} : Im(u) \neq 0, f(u) \in \mathbb{R}\}. \tag{14}$$

On the real plane Oxy, we associate the axis Ox with the real part, and Oy with the imaginary part of the complex number $u = a + bi$. One can easy prove next

Lemma 7. The carrier set N_f of the polynomial $f(x)$ coincides with the graph of the multivalued mapping

$$y = f_N(x) = \pm\sqrt{3x^2 + 2a_1\cdot x + a_2} = \pm\sqrt{f'(x)}. \tag{15}$$

The domain of $f_N(x)$ coincides with the projection of the set S_f onto the axis Ox. It is easy to see that the conjugate function $f_N(x)$ has two slope asymptotes

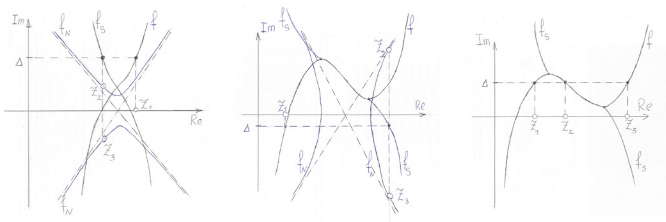
$$y = \sqrt{3}x + \frac{a_1}{\sqrt{3}}, y = -\sqrt{3}x - \frac{a_1}{\sqrt{3}}. \tag{16}$$

The slope of theese asymptotes is $\pm60^o$ with respect to Ox.

7. ALGORYTHM FOR THE EXACT GEOMETRIC SEARCH OF ALL ROOTS OF A CUBIC EQUATION

For simplicity of calculations, we shall assume that the elder coefficient of cubic equation $a_0 = 1$.

(1) Choose an arbitrary $\Delta\in\mathbb{R}$.
(2) Find the intersection of the graph $y = f(x)$ and the line $y = \Delta$. There are two possible cases:
    (a) The intersection consists of 3 points, of which two or three can coincide when the function graph and the line are touched. Then the abscissas z_1, z_2, z_3 of these points are obviously real roots (possibly multiples) of equation (2) (Fig. 4c). The algorithm is complete.
    (b) The intersection consists of one point. The abscissa z_1 of this point is the unique real root of equation (2). To find the two complex roots, go to step 3.



a) b) c)

Fig. 4. a) Always one real and two complex roots, b)one real and two complex roots, c)three real roots.

(3) Find the intersection point of the line $y = \Delta$ and the gluing set S_f. This point is unique. Its abscissa a is the real part of the required pair of complex conjugate roots.
(4) Find the intersection of the vertical line $x = a$ and the carrier set N_f, which lies on graph of f_N. This intersection consists of two different points. The ordinates of these points are equal to $+b$ and $-b$, where $b \geq 0$ and are imaginary components of the required pair of complex-conjugate roots of equation (2).
(5) Thus, we have obtained two real numbers a and b, which yield complex conjugate roots of equation (2) (Fig. 4a, 4b). The algorithm is complete.

8. THE GENERAL CASE OF A POLYNOMIAL OF ORDER N

Consider a polynomial of arbitrary order n with $a_0 = 1$.

$$P_n(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n.$$

Replace $x = a+bi$ and group the real and imaginary parts of the polynomial P_n

$$P_n(a + bi) = U(a, b) + V(a, b)i.$$

Equate $V(a, b) = 0$ and express b by a: $b = g(a)$, where the domain of the real function $g(\cdot)$ may be less then $\mathbb{R}$.

8.1 Finding carrier set

We consider two cases of degree n.

The degree n is even In this case, the polynomial V has an odd degree $n - 1$ with respect to the variables a and b, and the variable b is a factor. Therefore $V(a, b) = b \cdot V_1(a, b)$, where the degree V_1 of b is $n - 2$ and V_1 depends on b^2. In this way, $V_1(a, b) = V_2(a, b^2)$ and after the change $b' = b^2$ the polynomial $V_2(a, b')$ has degree $(n - 2)/2$ with respect to variable b'. Taking into account the possibility of using the Torricelli formula for solving the fourth-degree equation, the equation $V_2(a, b') = 0$ can be solved with respect to b' for the powers $(n-2)/2 \leq 4$, from which $n \leq 10$. As a result, we obtain the exact analytic representation for $b = g(a)$ for even $n \leq 10$, that is, for $n = 4, 6, 8, 10$.

The degree n is uneven In this case, the polynomial V has degree $n - 1$ with respect to a and degree n with respect to b. In addition, also $V(a, b) = b \cdot V_1(a, b) = b \cdot V_2(a, b')$, where $b' = b^2$ and the polynomial V_2 with respect to b' has degree $(n - 2)/2$. Therefore, we can obtain an analytic representation for $b = g(a)$ for $n \leq 9$, that is, for $n = 3, 5, 7, 9$.

Taking into account both cases, the function $g(a)$ can be obtained for all integers n from 3 to 10. Since $g(a)$ is the complex part of the root for the known part of a, the function $g(a) = f_N(x)$, that is, $g(\cdot)$ is the carrier for the polynomial $P_n(x)$.

8.2 Finding the conjugate function

The function $g(\cdot)$ is multivalued and its graph consists of several branches. We substitute the expression of some branch $b = g(a)$ into the real part $U(a, b)$ of the polynomial. We get $h(a) = U(a, g(a))$. Obviously, the domains of the functions $U(a)$ and $h(a)$ coincide. The function $U(a)$ is also multivalued, and a one-to-one correspondence can be established between the branches of both functions. The graph of the function $h(a)$ forms a gluing set of the polynomial $P_n(x)$ and on its domain of definition the function $h(a)$ coincides with the conjugate function $f_S(x)$. The real parts a of all complex roots of equations (9) can be found graphically from the equation $h(a) = \Delta$.

9. OPTIMIZATION OF COMPLEX POLYNOMIALS

An important class of smooth functions are polynomial functions. The optimization of polynomials is studied in (Marshall (2013), Ha (2007)). Consider using of complex numbers for creating criteria for the optimality of polynomials of one variable.

As is known, any point of the extremum x^* of a smooth function $f(x)$ is a stationary point for this function, that is, $f'(x^*) = 0$. The converse is not true. A stationary point that is not an extremum is called a false extremum.

The polynomial $f(x)$ of n-th order has exactly $n-1$ stationary points. Consider the perturbed polynomial $f(x, \alpha) = f(x) + \alpha x$, where α is a sufficiently small modulus. Obviously, $f(x, 0) = f(x)$. Let $x^*, x^*(\alpha)$ are the close stationary points of the polynomials $f(x), f(x, \alpha)$. Consider the behavior of the point $x^*(\alpha)$ for small variations of the parameter α.

Let x^* be a local minimum point of the function $f(x)$. Then the derivative $f'(x)$ increases in a small neighborhood of the point x^* and $f'(x^*) = 0$. By the implicit function theorem for small α, the equation $f'(x, \alpha) = f'(x) + \alpha = 0$ is solvable and has a solution $x^*(\alpha)$. The point $x^*(\alpha)$ is the local minimum of the function $f(x, \alpha)$ and increases with increasing α in some small range $(-\alpha_0, \alpha_0), \alpha_0 > 0$.

If x^* is a local maximum point of $f(x)$, then for small α, the local maximum point $x^*(\alpha)$ of $f(x, \alpha)$ exists and decreases with increasing α.

As a rule, the concept of a local extremum x^* is defined in terms of the values of the function $f(x)$ in the local neighborhood of the point x^*. Instead, we suggest using stationarity property of the point x^* and its behavior under perturbation of the function $f(x)$ by linear functions $\alpha \cdot x$ for small α.

Assume that x^* is a false extremum. Then x^* is also a stationary point, that is, $f'(x^*) = 0$ and $f(x)$ is monotonic in a neighborhood of x^*. Let $f(x)$ is increasing. Then for $\alpha < 0$, two local extrema $x_1^*(\alpha) < x^* < x_2^*(\alpha)$ of the function $f(x, \alpha)$ appear in a small neighborhood of x^*. For $\alpha > 0$, in this neighborhood the real stationary points of $f(x, \alpha)$ disappear and so function $f(x, \alpha)$ has no local extrema.

Proposition 8. . A complex-valued function $x^*(\alpha)$ of a real argument α is called the trajectory of the stationary point x^*.

The function $x^*(\alpha)$ is defined in a neighborhood of zero and $x^*(0) = x^*$. Ther are thre types of trajectories of stationary point:

(1) real, i.e. for all α point $x^*(\alpha)$ is real;
(2) complex, i.e. for all $\alpha \neq 0$ point $x^*(\alpha)$ is complex. Complex trajectories are grouped into complex conjugate pairs.
(3) semi-real, i.e. $x^*(\alpha)$ is real only for $\alpha > 0$ or for $\alpha < 0$.

We now give the definition of a complex local extremum of the real polynomial $f(x)$.

Proposition 9. . Let x^* be a complex or real stationary point of the real polynomial $f(x)$. Let the parameter α varies in a small range of $(-\alpha_0, \alpha_0), \alpha_0 > 0$ and $x^*(\alpha)$ - stationary point of a real polynomial $f(x, \alpha)$ in some small neighborhood of the point x^*. If the real part of $Re\ x^*(\alpha)$ decreases with increasing α, then x^* will be called the point of local complex minimum. If the real part $Re\ x^*(\alpha)$ increases with increasing α, then x^* will be called the point of local complex maximum.

Let us formulate a sufficient criterion for optimality.

Theorem 10. . Let x^* be a real stationary point with multiplicity k. Then k trajectories pass through x^*. There are two possible cases:

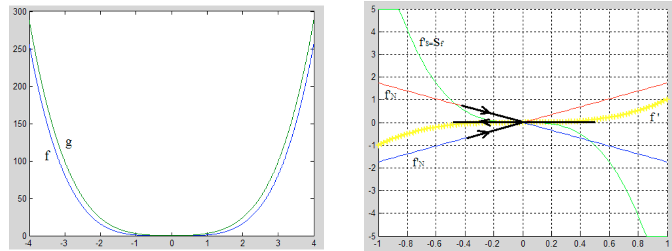
(1) There is exactly one real path among all all of the trajectories. Then x^* is a real extremum point. The trajectory is monotonous. If the trajectory increases, then x^* is a local maximum. If the trajectory decreases, then x^* is a local minimum.
(2) There are exactly two semi-real trajectories among all of the trajectories and their real halves have different monotonicity. In this case x^* is not extremum.

9.1 Example

We consider two polynomials $f(x) = x^4, g(x) = (x^2+1)^2 - 1$. Surprisingly, both graphics are similar to each other. They are parallel lines at some distance from zero (Fig. 5a). Both functions have a minimum at the point $x_0 = 0$. For a function $g(x)$ this minimum has multiplicity 1. For the function $f(x)$, the situation is different. The multiplicity of the minimum point is three and three trajectories intersect in it. Three multiple stationary points are the roots of the cubic equation $x^3 = 0$. Our algorithm allows us to construct these trajectories: one trajectory of the real minimum $x_0 = 0$ and two complex conjugate trajectories of the complex maximum $x_0 = 0$ (Fig. 5b).

9.2 Numerical calculations

Consider the polynomial
$f(x) = 1/6x^6 - 9/5x^5 + 32/4x^4 - 56/3x^3 + 48/2x^2 - 16x$. The derivative is equal to $f'(x) = (x-1)(x-2)^4$. Thus, the polynomial $f(x)$ has five stationary points: local minimum $x^* = 0$ with multiplicity 1, and false minimum $x^* = 0$ with multiplicity 4 (Table 1).



a) b)

Fig. 5. Two parabolas of the 4th order with multiplicity 1 (f) and 3 (g): a) Graphs of f and g, b) The minimum point with multiplicity 3

Table 1. Trajectories of stationary points on the complex plane for α = from -0.02 to 0.02 with step 0.01

x_1	x_2	x_3	x_4	x_5
1.02	1.56	2.34	2.03 - 0.36i	2.03 + 0.36i
1.01	1.64	2.29	2.02 - 0.31i	2.02 + 0.31i
1.00	2.00	2.00	2.00	2.00
0.99	2.22 - 0.20i	2.22 + 0.20i	1.78 - 0.25i	1.78 + 0.25i
0.98	2.26 - 0.23i	2.26 + 0.23i	1.74 - 0.30i	1.74 + 0.30i

10. TECHNICAL APPLICATIONS

Complex functions and their graphs play an important role in the technical sciences. For example, complex transfer function of an automatic control object contains information about the stability of the object. Criteria for the stability of closed control systems use amplitude-frequency characteristics, which are projections of the four-dimensional graph of the transfer function on two-dimensional planes.

A visual representation on the real plane of the set of n roots of the characteristic polynomial is called a root hodograph. It simplifies the synthesis of stable closed systems. Earlier (Uderman (1972)-Bendrikov (1955)), the root hodographs were drawn in an approximate way. Our algorithm gives an exact method for their construction.

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